On Schwarzschild's Stability Criterion

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(Z. Naturforsch. **26 a**, 1992—1994 [1971]; received 20 September 1971)

The stability of a gas in an external gravitational field Φ is investigated for arbitrary initial conditions. It is shown that Schwarzschild's entropy criterion $ds/d\Phi > 0$ is both necessary and sufficient for the spatial quadratic mean of a linear disturbance to be bounded in time. The variational principle usually applied is not always sufficient for stability.

If the atmosphere of a resting planet is treated as an ideal compressible fluid, the conservation equations of mass, momentum, and energy are

$$d\varrho/dt = -\varrho \nabla \cdot \boldsymbol{v}\,, \tag{1}$$

$$\varrho\, d\boldsymbol{v}/dt = -\varrho \nabla \Phi - \nabla p\,, \tag{2}$$

$$ds/dt = 0 \tag{3}$$

where s is the entropy per unit mass and the other symbols have their usual meaning. The gravitation of the atmosphere is neglected, so that the gravitational potential Φ can be considered as a given function of the position vector $\boldsymbol{x}$. Eliminating the entropy with the aid of some equation of state we obtain

$$\frac{dp}{dt} = c^2 \frac{d\varrho}{dt} + \mu \frac{ds}{dt} = -\varrho\, c^2 \nabla \cdot \boldsymbol{v} \tag{4}$$

where

$$c^2 = (\partial p/\partial \varrho)_s$$

is the speed of sound and

$$\mu = (\partial p/\partial s)_\varrho$$

is a quantity which for most gases is positive (e. g. for an ideal gas with constant specific heats c_v, c_p we have $\mu = p/c_v$).

For the state of a static equilibrium ($\boldsymbol{v} \equiv 0$, $\partial/\partial t \equiv 0$) characterized by the density ϱ_0 and the pressure p_0 Eqs. (1), (2), (3) reduce to

$$-\varrho_0 \nabla \Phi - \nabla p_0 = 0 \tag{5}$$

with the solution

$$\varrho_0 = -p_0{}' \tag{6}$$

where the prime denotes the derivative with respect to Φ.

The pressure distribution $p_0(\Phi)$ is a positive and decreasing function, but otherwise arbitrary.

In order to decide which of the equilibrium distributions $p_0(\Phi)$ are stable, Eqs. (1), (2), (4) are linearized in the disturbance quantities $\boldsymbol{v}$, $\tilde{\varrho}$, $\tilde{p}$, giving

$$\partial \tilde{\varrho}/\partial t = -\boldsymbol{v} \cdot \nabla \varrho_0 - \varrho_0 \nabla \cdot \boldsymbol{v}\,, \tag{7}$$

$$\varrho_0\, \partial \boldsymbol{v}/\partial t = -\tilde{\varrho} \nabla \Phi - \nabla \tilde{p}\,, \tag{8}$$

$$\partial \tilde{p}/\partial t = -\boldsymbol{v} \cdot \nabla p_0 - c^2 \varrho_0 \nabla \cdot \boldsymbol{v}\,. \tag{9}$$

Writing the initial conditions in the form

$$\begin{aligned} \boldsymbol{v}(\boldsymbol{x}, 0) &= \boldsymbol{v}^*\,, \\ \tilde{\varrho}(\boldsymbol{x}, 0) &= \varrho^*\,, \\ \tilde{p}(\boldsymbol{x}, 0) &= p^*\,, \end{aligned} \tag{10}$$

we can formulate the following stability problem: under what conditions on $p_0(\Phi)$ have Eqs. (7) – (10) solutions such that the spatial averages $\frac{1}{2}\langle \varrho_0 \boldsymbol{v}^2 \rangle$, $\langle \tilde{\varrho}^2 \rangle$, and $\langle \tilde{p}^2 \rangle$ remain bounded in time?

Introducing the displacement vector $\boldsymbol{\xi}$ by

$$\boldsymbol{v} = \partial \boldsymbol{\xi}/\partial t \tag{11}$$

$$\boldsymbol{\xi}(\boldsymbol{x}, 0) = 0 \tag{12}$$

one can integrate Eqs. (7) and (9) in time

$$\tilde{\varrho} = -c\, \varrho_0{}'\, \eta - c^{-1} \varrho_0\, \zeta + \varrho^*, \tag{13}$$

$$\tilde{p} = -c(p_0{}'\, \eta + \varrho_0\, \zeta) + p^* \tag{14}$$

where

$$c\,\eta = \boldsymbol{\xi} \cdot \nabla \Phi\,, \tag{15}$$

$$c^{-1} \zeta = \nabla \cdot \boldsymbol{\xi}\,. \tag{16}$$

Equations (13) and (14) differ from those used in the "energy principle"[1–4], where Eq. (12) and the

[1] R. FJÖRTOFT, Geof. Publ. **17** [1950].

[2] A. ELIASSEN, Handbuch der Physik, Bd. 48, Springer-Verlag, Berlin 1957.

[3] S. CHANDRASEKHAR, Astrophys. J. **139**, 664 [1964].

[4] S. KANIEL and A. KOVETZ, Phys. Fluids **10**, 1186 [1967].

terms ϱ^*, p^* in Eqs. (13), (14) are omitted so that the initial conditions on $\tilde{\varrho}$, $\tilde{p}$ are expressed by initial conditions on $\boldsymbol{\xi}$. This, however, is not general case, as one sees by multiplying Eq. (16) by $|\nabla\Phi|^{-1}$ and integrating over a surface $\Phi=$ const

$$\oint\!\!\oint \frac{\zeta}{c\,|\nabla\Phi|}\,\mathrm{d}S = \oint\!\!\oint \frac{\nabla\cdot\boldsymbol{\xi}}{|\nabla\Phi|}\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}\Phi}\oint\!\!\oint \boldsymbol{\xi} \cdot \frac{\nabla\Phi}{|\nabla\Phi|}\,\mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}\Phi}\oint\!\!\oint \frac{c\,\eta}{|\nabla\Phi|}\,\mathrm{d}S\,. \tag{17}$$

Since $\tilde{\varrho}(\boldsymbol{x},0)$, $\tilde{p}(\boldsymbol{x},0)$ are arbitrary, $\eta(\boldsymbol{x},0)$, $\zeta(\boldsymbol{x},0)$ are also arbitrary, which contradicts Equation (17).

Substituting Eqs. (11), (13), and (14) into Eq. (8) we find with the equilibrium condition (6)

$$\varrho_0\,\partial^2\boldsymbol{\xi}/\partial t^2 = \mathcal{F}\,\boldsymbol{\xi} - \varrho^*\,\nabla\Phi - \nabla p^* \tag{18}$$

where $\mathcal{F}$ is the "stability operator"

$$\mathcal{F}\,\boldsymbol{\xi} = -(c\,\varrho_0{}'\,\eta + c^{-1}\,\varrho_0\,\zeta)\,\nabla\Phi + \nabla c\,\varrho_0(\eta-\zeta). \tag{19}$$

Under certain boundary conditions, for instance if the normal component of $\boldsymbol{\xi}$ vanishes on the ground and at a great height, the operator $\mathcal{F}$ is symmetric

$$\langle\boldsymbol{\xi}^{(1)}\cdot\mathcal{F}\,\boldsymbol{\xi}^{(2)}\rangle = \langle\boldsymbol{\xi}^{(2)}\cdot\mathcal{F}\,\boldsymbol{\xi}^{(1)}\rangle\,. \tag{20}$$

Assuming (20) we multiply Eq. (18) by $\partial\boldsymbol{\xi}/\partial t$, take the space average, and find after integrations by part for the kinetic energy K of the disturbance

$$K = -W - \langle L\rangle + K_0 \tag{21}$$

where

$$K = \tfrac{1}{2}\,\langle\varrho_0\,\boldsymbol{v}^2\rangle\,,\quad K_0 = \tfrac{1}{2}\,\langle\varrho_0\,\boldsymbol{v}^{*2}\rangle\,,$$

$$L(\eta,\zeta) = c\,\varrho^*\,\eta + c^{-1}\,p^*\,\zeta\,,$$

and the functional

$$W = \tfrac{1}{2}\,\langle\boldsymbol{\xi}\cdot\mathcal{F}\,\boldsymbol{\xi}\rangle = \tfrac{1}{2}\,\langle Q\rangle \tag{22}$$

with $$Q(\eta,\zeta) = -\varrho_0{}'\,c^2\,\eta^2 - 2\,\varrho_0\,\eta\,\zeta + \varrho_0\,\zeta^2 \tag{23}$$

is the so-called energy integral. The equation of state and the equilibrium condition (6) yield

$$p_0{}' = -\varrho_0 = c^2\,\varrho_0{}' + \mu\,s_0{}'. \tag{24}$$

Therefore

$$Q = \varrho_0(\eta-\zeta)^2 + \mu\,s_0{}'\,\eta^2. \tag{25}$$

Suppose now that everywhere

$$\mu\,s_0{}' = -c^2\,\varrho_0{}' - \varrho_0 > 0\,. \tag{26}$$

The quadratic form $Q(\eta,\zeta)$ is then positive definite and therefore $\tfrac{1}{2}\,Q + L$ is bounded from below.

$$\tfrac{1}{2}\,Q + L \geqq -C_1^2.$$

Consequently, the kinetic energy of the disturbance is bounded

$$K \leqq K_0 + C_1^2.$$

Furthermore, from relation (21) we can derive the estimate

$$K_0 \geqq K_0 - K = \langle\tfrac{1}{2}\,Q + L\rangle \geqq \tfrac{1}{2}\,W - C_2^2$$

where

$$\tfrac{1}{4}\,Q + L \geqq -C_2^2.$$

This means that W is also bounded from above

$$0 \leqq W \leqq C_3 = 2(K_0 + C_2^2)\,. \tag{27}$$

The quadratic mean of Eq. (13) yields

$$\langle\tilde{\varrho}^2\rangle = \langle(-c\,\varrho_0{}'\,\eta - c^{-1}\,\varrho_0\,\zeta + \varrho^*)^2\rangle \leqq 2\langle(c\,\varrho_0{}'\,\eta + c^{-1}\,\varrho_0\,\zeta)^2 + \varrho^{*2}\rangle.$$

Then using (26) and (27) we find

$$\langle(c\,\varrho_0{}'\,\eta + c^{-1}\,\varrho_0\,\zeta)^2\rangle \leqq \langle\varrho_0{}'(c^2\,\varrho_0{}'\,\eta^2 + 2\,\varrho_0\,\eta\,\zeta - \varrho_0\,\zeta^2)\rangle \leqq 2\,W\max(-\varrho_0{}') \leqq 2\,C_3\max(-\varrho_0{}')\,.$$

Correspondingly, from Eq. (14)

$$\langle\tilde{p}^2\rangle \leqq 2\langle c^2(p_0{}'\,\eta + \varrho_0\,\zeta)^2 + p^{*2}\rangle\,.$$

Here we find the estimate

$$\langle c^2(p_0{}'\,\eta + \varrho_0\,\zeta)^2\rangle \leqq \langle\varrho_0(\eta-\zeta)^2\rangle\max(\varrho_0\,c^2) \leqq 2\,C_3\max(\varrho_0\,c^2)\,.$$

Hence condition (26) is sufficient for stability.

For the case that the left-hand side of (26) is somewhere negative KANIEL and KOVETZ [4] have shown by an eigenfunction expansion that exponentially growing solutions exist. However, it is easier to apply the following theorem [5]: If a vector field $\boldsymbol{\xi}$ satisfying the boundary conditions can be constructed in such a way that the functional W is negative, there then always exists a disturbance which grows exponentially in time.

Choosing

$$\zeta = \eta \tag{28}$$

and η non-zero only where the left-hand side of (26) is negative, Q and hence W can be made negative.

In order to construct a vector field $\boldsymbol{\xi}$ that satisfies Eq. (28), let us introduce the coordinate system Φ, ψ, χ where $0 \leqq \psi \leqq \pi$ behaves like a spherical pole distance. Then Eq. (28) reads

$$c\,D[\partial(c\,D^{-1}\,\eta)/\partial\Phi + \partial\alpha/\partial\psi + \partial\beta/\partial\chi] = \eta \tag{29}$$

[5] G. LAVAL, C. MERCIER, and R. PELLAT, Nucl. Fusion 5, 156 [1965].

where

$$D = (\nabla \psi \times \nabla \chi) \cdot \nabla \Phi$$

is the functional determinant and

$$D\,\alpha = \boldsymbol{\xi} \cdot \nabla \psi\,, \quad D\,\beta = \boldsymbol{\xi} \cdot \nabla \chi$$

are the counter-variant components of $\boldsymbol{\xi}$ in the ψ, χ direction. A solution of Eq. (29) which corresponds to a unique, continuously differentiable vector field $\boldsymbol{\xi}$ is, for instance,

$$\begin{aligned} \eta &= D\,f(\Phi)\,h'(\psi)\,, \\ \alpha &= [c^{-1} f - (c\,f)']\,h\,, \\ \beta &= 0 \end{aligned} \qquad (30)$$

where $h(\psi)$ is a function which vanishes at least quadratically at the poles $\psi = 0$, $\psi = \pi$, and $f(\Phi)$ is non-zero only where the heft-hand side of (26) is negative.

It now remains to discuss the case that there is an interval where $\mu\,s_0'$ vanishes identically, while it is non-negative outside this interval. For this case the variational principle asserts that the system is stable. However, it is easy to construct solutions such that the kinetic energy is unbounded.

Suppose that

$$\mu\,s_0' = -c^2\,\varrho_0' - \varrho_0 = 0 \quad \text{for} \quad \Phi_1 \leqq \Phi \leqq \Phi_2 \qquad (31)$$

and that $\tilde{p}$ and $\tilde{\varrho}$ are time-independent. Then Eqs. (7) and (9) are multiples of each other, and if the velocity is growing linearly in time

$$\boldsymbol{v} = \boldsymbol{w}(\boldsymbol{x})\,t \qquad (32)$$

the disturbance equations reduce to

$$\begin{aligned} \varrho_0\,\boldsymbol{w} &= -\tilde{\varrho}\,\nabla \Phi - \nabla \tilde{p}\,, \\ 0 &= \nabla \cdot (\tilde{\varrho}\,\nabla \Phi) + \Delta \tilde{p}\,. \end{aligned} \qquad (33)$$

Equation (33) is a scalar equation for the two scalar unknowns $\tilde{\varrho}$ and $\tilde{p}$, while Φ is a given function in space. If we impose the boundary conditions

$$\tilde{\varrho} = \tilde{p} = \nabla \tilde{p} \cdot \nabla \Phi = 0 \quad \text{for} \quad \Phi = \Phi_1 \quad \text{and} \quad \Phi = \Phi_2 \qquad (34)$$

then the disturbance vanishes outside the interval $\Phi_1 \leqq \Phi \leqq \Phi_2$. For the case that Φ has spherical symmetry $[\Phi = \Phi(r)]$ the solutions of Eqs. (33) and (34) can be explicitly written down in spherical coordinates r, ϑ, φ:

$$\tilde{p} = G'(r)\,Y_n(\vartheta, \varphi)\,, \quad \tilde{\varrho} = g(r)\,Y_n(\vartheta, \varphi)\,,$$
$$g\,\Phi' + G'' - n(n+1)\,r^{-2}\,G = 0, \quad n = 1, 2, 3, \ldots\,.$$

Here $Y_n(\vartheta, \varphi)$ are the spherical harmonics which can be expressed by associated Legendre polynomials and $G(r)$ is an arbitrary function which together with its first and second derivatives vanishes at the boundary.

This kind of instability corresponds to the magnetohydrodynamic case, where one can also show [6] that the variational principle is not, in general, sufficient for stability.

Acknowledgement

The author would like to thank his colleagues of the Theory Division of the Institute of Plasma Physics for many interesting and stimulating discussions. — This work was performed under the terms of the agreement on association between the Max-Planck-Institut für Plasmaphysik and EURATOM.

[6] D. Lortz and E. Rebhan, Phys. Letters **35 A**, 236 [1971].